Stability properties of cellular automata

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Outline

1. Outline
2. Introduction
3. Stability theory of continuous systems
4. Characterizing CA dynamics
   - Incentives and issues
   - Stability assessment
   - Jacobian-based measure
5. Simulation study and results
6. The story continues
7. Ongoing work

Question

What happens to a dynamical system in the end?

- Steady-state?
- Periodicity?
- Chaos?

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Lyapunovian stability

Methodology

- Evolution is displayed in phase space

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- Fate of nearby trajectories is tracked as time elapses

Zero Lyapunov exponent

Lyapunovian stability

Methodology

- Evolution is displayed in phase space
- Fate of close trajectories is tracked as time elapses

Negative Lyapunov exponent

Lyapunovian stability

Methodology

- Evolution is displayed in phase space
- Fate of nearby trajectories is tracked as time elapses
- Continuous dynamical systems: ‘nearby’ is well defined
- Discrete dynamical systems: notion of ‘nearby’ gets blurred

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CA dynamics

- Overwhelming variety of evolved spatio-temporal patterns

- Unexpected since CA are inherently simple
- Caught the attention of numerous researchers

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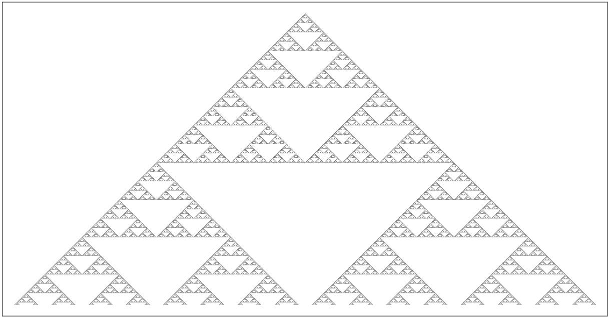

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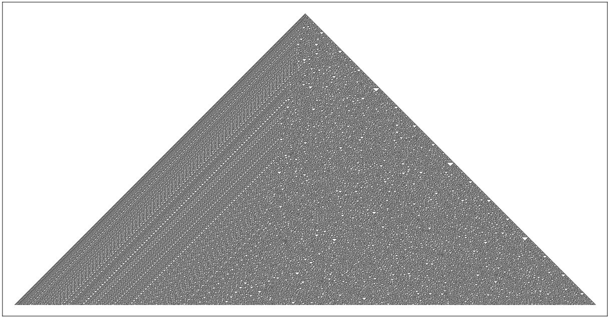

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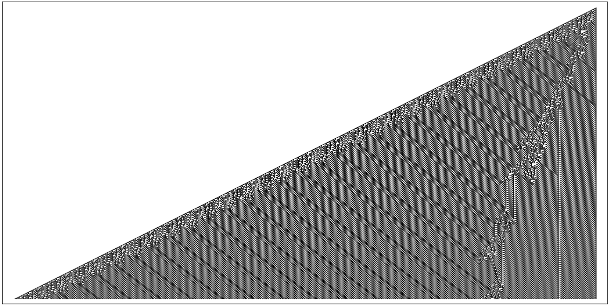

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How to grasp CA dynamics?

- Wolfram's classification scheme
  - Four behavioral classes
  - Presumption: every CA can be assigned to one or another class
  - Involves a trace of subjectivity (Class 3 vs. class 4)
  - Time-consuming, manual classification
- Towards a quantitative approach
  - measures grasping CA dynamics
  - do not require a personal opinion: objective
  - can be calculated automatically
  - many have been proposed:
    - entropies
    - Hamming distance
    - Langton parameter
    - Lyapunov exponents

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    - **Lyapunov exponents**

Lyapunov exponents

- Quantifies the speed of divergence of trajectories in phase space originating from a perturbation of one of the cells' state
- Suggested by Wolfram, formalized by Shereshevsky (1991)
- 1D CA: directional Lyapunov exponents
- Higher-dimensional CA: directionality becomes blurred
- Way out: non-directional Lyapunov exponent (Bagnoli et al., 1992)
  - a theoretical upper bound can be derived
  - based upon a Jacobian-based measure
  - restricted to CA based upon regular tessellations

Nearby trajectories

- $S = \{0, 1\}$
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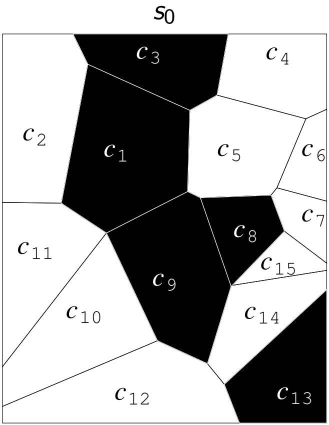
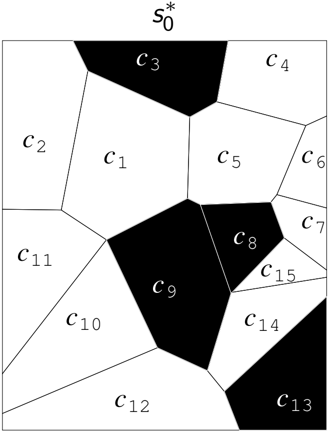
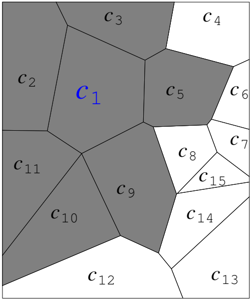




Accumulation of defects

0th time step



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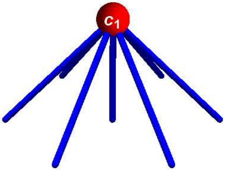
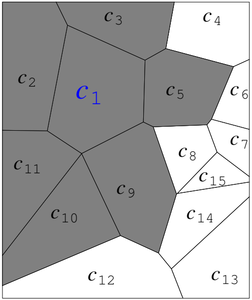
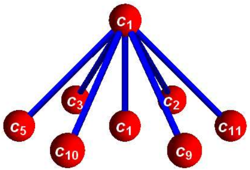
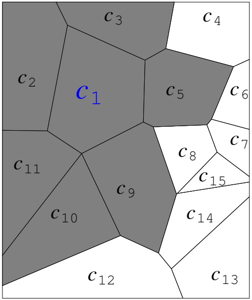




Accumulation of defects

1st time step





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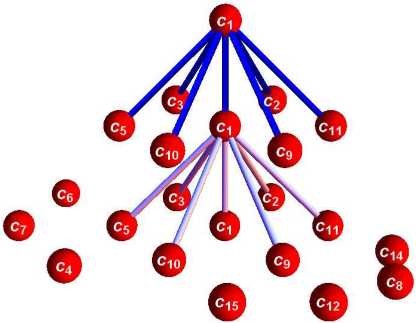

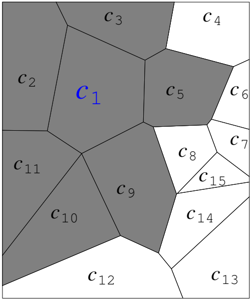

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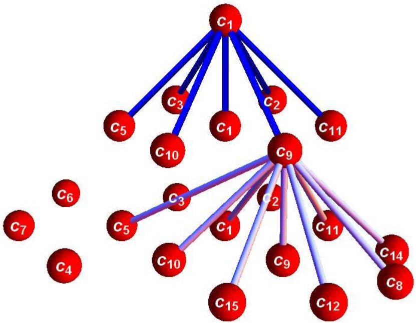

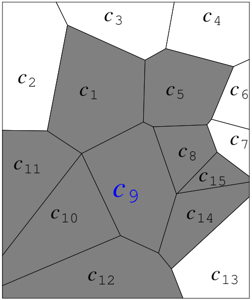

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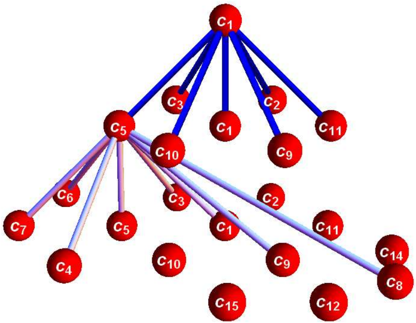

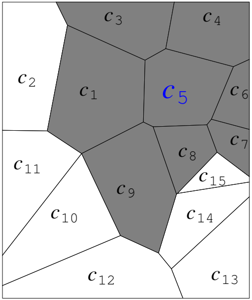

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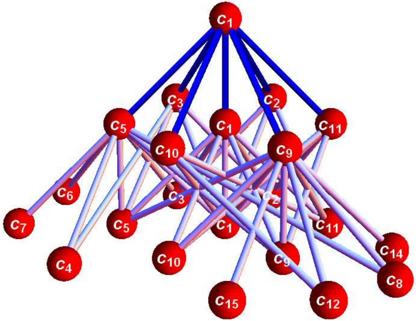
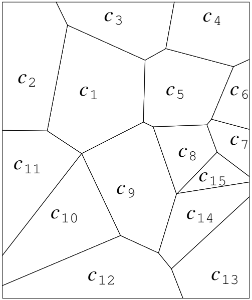




Lyapunov exponents of two-state cellular automata

Maximum Lyapunov exponent (MLE):

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log\left(\frac{\epsilon_t}{\epsilon_0}\right)$$

with

$$\epsilon_t = |\{i \mid s(c_i, t) \neq s^*(c_i, t)\}|$$

- Quantify the rate with which the number of defects increases
- A theoretical upper bound on the Lyapunov exponent (λ) can be derived
- Stability classes
  - $\lambda = -\infty$: superstable
  - $\lambda = 0$: stable
  - $\lambda > 0$: unstable
- Applicable to any kind of two-state dynamical system

Input sensitivity

- Question: how sensitive is ϕ_i to an alteration of its inputs?
- Addressed for regular CA by Bagnoli et al. (1992)
- Can be reformulated more generally for variable $|N(c_i)|$
- Formulation valid for (ir)regular CA, graph CA,. . . :

$$\mu(t) = \frac{1}{|\mathcal{T}^*|} \sum_{c_i} \frac{1}{|N(c_i)|} \sum_{j=1}^{|N(c_i)|} J_{ii_j}$$

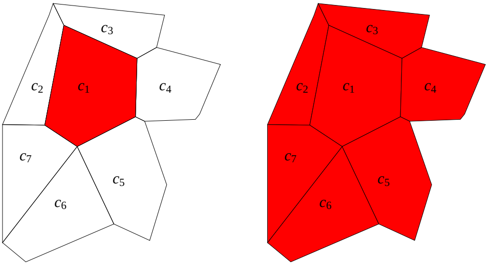
- J: $|\mathcal{T}^*| \times |\mathcal{T}^*|$ Jacobian
- $J_{ij} = 1$ iff an alteration of $s(c_j, t)$ (Boolean complement) affects $s(c_i, t+1)$
- Analysis based on its geometric mean $\bar{\mu} \in [0, 1]$ after T time steps

An upper bound on the MLE of irregular CA

- The number of cells that is infected by a defective c_i during one time step:

$$\frac{\epsilon_{t+1}}{\epsilon_t}$$

- This number is upper bounded by $|N(c_i)|$



- A damaged c_i gives rise to at most $|N(c_i)|$ additional defects
- Implication: $\bar{\mu} = 1$ (and vice versa)

An upper bound on the MLE of irregular CA

- Irregular tessellations, graph CA: variable $|N(c_i)|$
- Mean-field approximation for $t \to \infty$: $|N(c_i)| \approx \overline{V}$
- Mean-field estimate of the upper bound on the MLE

$$\lambda_m = \log\left(\overline{V}\right)$$

- For a given $\bar{\mu}$:

$$\lambda_m(\bar{\mu}) = \log\left(\overline{V}\bar{\mu}\right)$$

since $\overline{V}\bar{\mu}$ gives the number of susceptible cells

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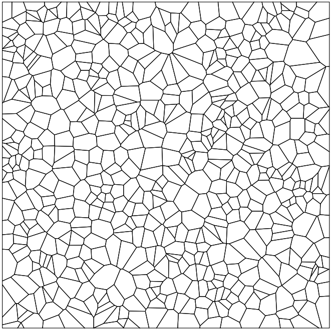
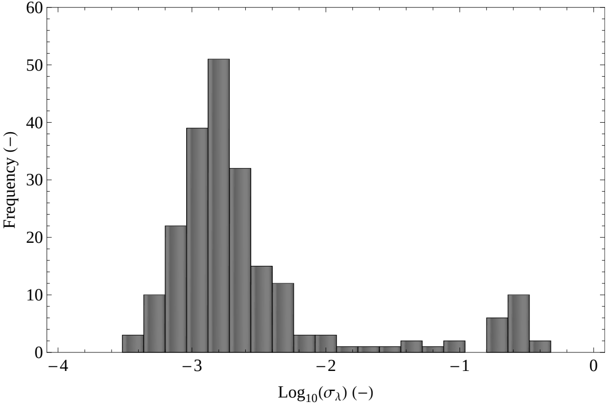
Simulation setup

- $\mathcal{T}^*$: Voronoi tessellation containing 675 cells
- Periodic boundary conditions
- λ averaged over an ensemble of initial perturbations
- CA family: 256 (2,7) irregular totalistic CA



Simulation results



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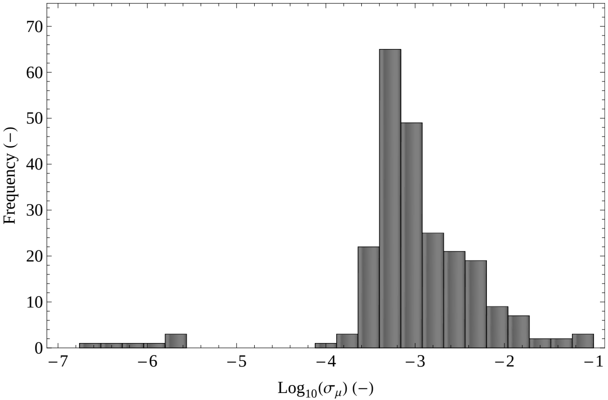

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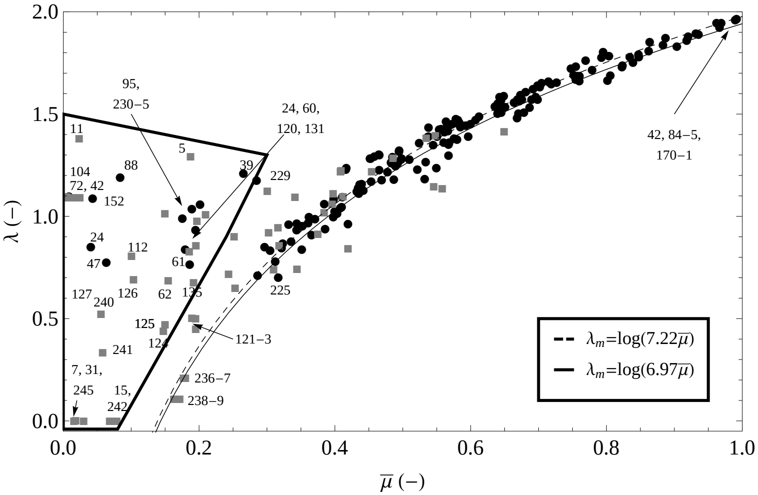

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Topological bifurcations

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  - stability can be strongly affected by the characteristics of the underlying topology
  - topological bifurcation points emerge

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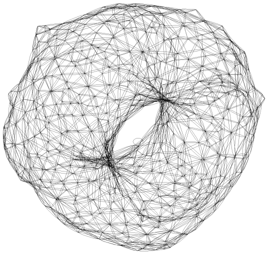

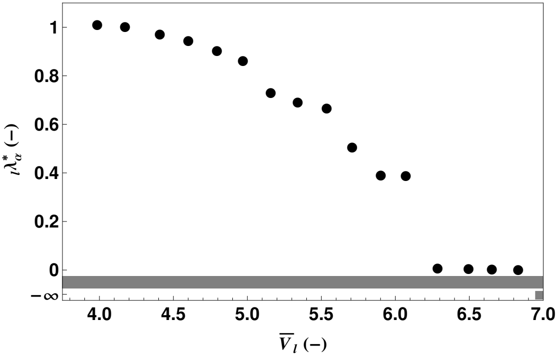

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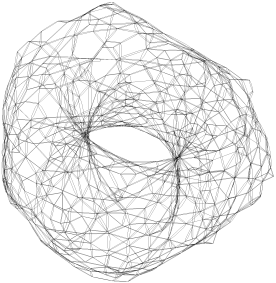

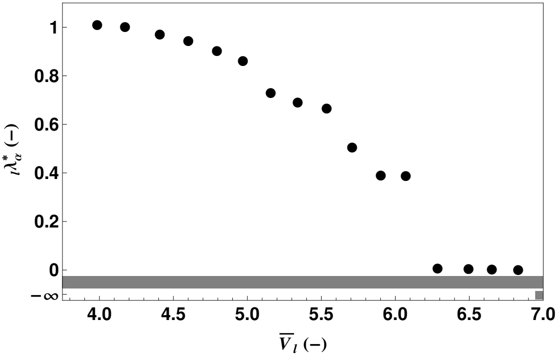

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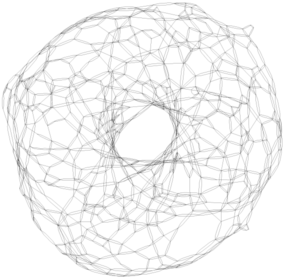
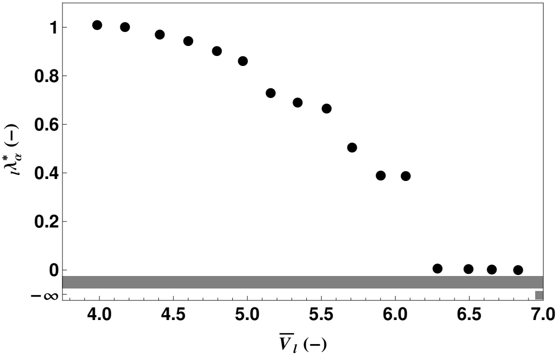
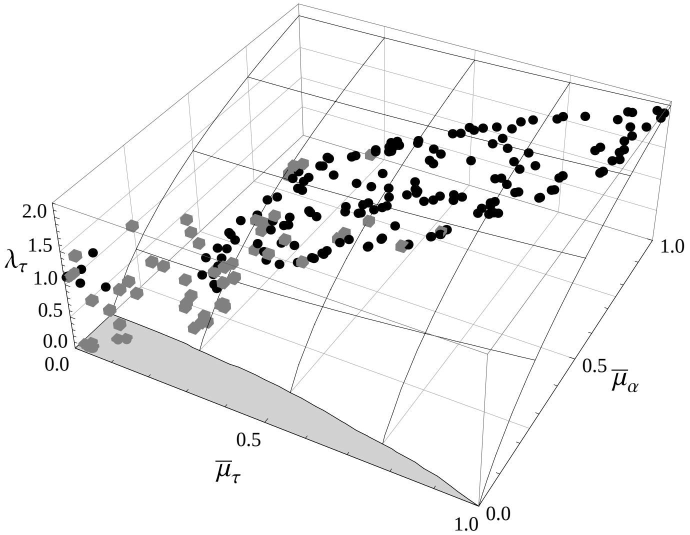
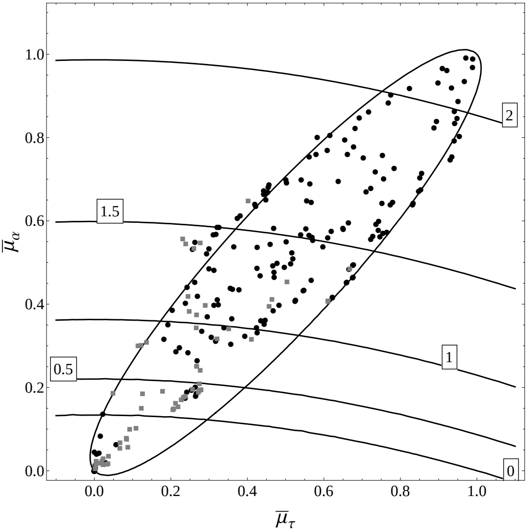




Topological Lyapunov exponents



Prerequisites for instability



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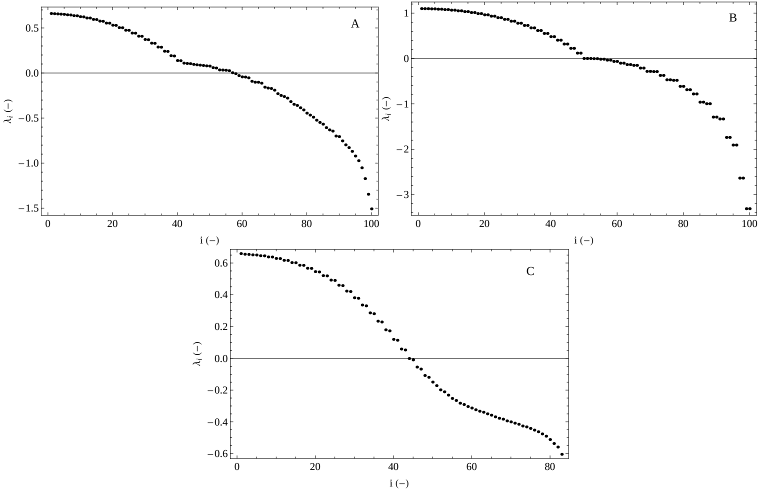
The full spectra



Conclusions

- Theoretical upper bound on the MLE can be derived
- The dynamical properties of a CA family can be summarized in a λ-$\bar{\mu}$ diagram
- Proposed methodology is applicable to any family of 2-state CA
- What is the meaning of the full spectrum?